ON THE SPECTRUM OF DEHN TWISTS IN QUANTUM TEICHMÜLLER THEORY

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ABSTRACT. The operator realizing a Dehn twist in quantum Teichmüller theory is diagonalized and continuous spectrum is obtained. This result is in agreement with the expected spectrum of conformal weights in quantum Liouville theory at $c > 1$. The completeness condition of the eigenvectors includes the integration measure which appeared in the representation theoretic approach to quantum Liouville theory by Ponsot and Teschner. The underlying quantum group structure is also revealed.

1. INTRODUCTION

The quantization problem of Teichmüller spaces of punctured surfaces is connected to quantum Chern–Simons theory with non-compact gauge groups (and thereby to 2+1-dimensional quantum gravity [1, 22]) and non-rational quantum conformal field theory in two dimensions. In particular, quantum Teichmüller theory is expected to describe the space of conformal blocks in quantum Liouville theory [21]. From the mathematical point of view this can be useful in three-dimensional topology, in particular for construction of new link and three-manifold invariants.

The Teichmüller spaces of punctured surfaces have been quantized in two different but essentially equivalent ways: as (degenerate) Poisson manifolds [8, 3], and as simplectic manifolds with Weil–Petersson simplectic structure [13]. The both approaches employ the real analytic coordinates closely related to the Penner parameterization of the decorated Teichmüller spaces [17]. The main outcome of the theory is the construction of a particular projective (infinite dimensional) representation of the mapping class groups of punctured surfaces. The projective factor has been shown to be related to the Virasoro central charge in quantum Liouville theory [14].

In this paper we describe the spectrum of Dehn twists in the quantum Teichmüller theory, derive an explicit formula for the associated with braidings R-matrix (square of which is a Dehn twist), and uncover the underlying infinite dimensional representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2))$. Those representations were considered in papers [20, 18, 19].

The paper is organized as follows. In section 2 we briefly remind the main properties of the non-compact quantum dilogarithm (closely related to the double sine function, see for example [9]). This is the main technical object entering practically all the constructions of the paper. In section 3 we describe the algebraic system underlying the quantum Teichmüller theory and its realization in terms of the quantum dilogarithm. Section 4 is a survey of the part of papers [13, 14] which describes the mapping class group representation in terms of the basic algebraic system of section 3. Solution of the Dehn twist spectral problem is described in section 5 (proof of Theorem 1 is to appear in a separate publication). Section 6

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contains derivation of the R-matrix and revealing the associated quantum group structure.

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2. Quantum dilogarithm

Let complex b have a nonzero real part $\Re b \neq 0$. The *non-compact quantum dilogarithm*, $\mathrm{e}_b(z)$, is defined by the integral formula [5]

$$\mathrm{e}_b(z) \equiv \exp\left(\frac{1}{4}\int_{\mathrm{i}0-\infty}^{\mathrm{i}0+\infty} \frac{e^{-\mathrm{i}2zw}\, dw}{\sinh(wb)\sinh(w/b)w}\right) \tag{1}$$

in the strip $|\Im z| < |\Im c_b|$, where

$$c_b \equiv \mathrm{i}(b+b^{-1})/2.$$

In particular, when $\Im b^2 > 0$, we have the following product formula:

$$\mathrm{e}_b(z) = (e^{2\pi(z+c_b)b}; q^2)_\infty / (e^{2\pi(z-c_b)b^{-1}}; \bar{q}^2)_\infty, \tag{2}$$

where

$$q \equiv e^{\mathrm{i}\pi b^2}, \quad \bar{q} \equiv e^{-\mathrm{i}\pi b^{-2}}.$$

Using the symmetry properties

$$\mathrm{e}_b(z) = \mathrm{e}_{-b}(z) = \mathrm{e}_{1/b}(z),$$

we assume in what follows

$$\Re b > 0, \quad \Im b \geq 0.$$

Function $\mathrm{e}_b(z)$ can be analytically continued in variable z to the entire complex plane as a meromorphic function with essential singularity at infinity, and with the following characteristic properties:

poles and zeros:

$$(\mathrm{e}_b(z))^{\pm 1} = 0 \ \Leftrightarrow \ z = \mp(c_b + m\mathrm{i}b + n\mathrm{i}b^{-1}),\ m, n \in \mathbb{Z}_{\geq 0}; \tag{3}$$

behavior at infinity: depending on the direction along which the limit is taken,

$$\mathrm{e}_b(z)\Big|_{|z|\to\infty} \approx \begin{cases} 1 & |\arg z| > \frac{\pi}{2} + \arg b; \\ e^{\mathrm{i}\pi z^2 - \mathrm{i}\pi(1+2c_b^2)/6} & |\arg z| < \frac{\pi}{2} - \arg b; \\ \frac{(\bar{q}^2;\bar{q}^2)_\infty}{\Theta(\mathrm{i}b^{-1}z;-b^{-2})} & |\arg z - \frac{\pi}{2}| < \arg b; \\ \frac{\Theta(\mathrm{i}bz;b^2)}{(q^2;q^2)_\infty} & |\arg z + \frac{\pi}{2}| < \arg b, \end{cases} \tag{4}$$

where

$$\Theta(z;\tau) \equiv \sum_{n\in\mathbb{Z}} e^{\mathrm{i}\pi\tau n^2 + \mathrm{i}2\pi z n}, \quad \Im\tau > 0;$$

inversion relation:

$$\mathrm{e}_b(z)\,\mathrm{e}_b(-z) = e^{\mathrm{i}\pi z^2 - \mathrm{i}\pi(1+2c_b^2)/6}; \tag{5}$$

functional equations:

$$\mathrm{e}_b(z - \mathrm{i}b^{\pm 1}/2) = (1 + e^{2\pi b^{\pm 1} z})\,\mathrm{e}_b(z + \mathrm{i}b^{\pm 1}/2); \tag{6}$$

unitarity: when b is either real or on unit circle,

$$(1 - |b|)\Im b = 0 \ \Rightarrow \ \overline{\mathrm{e}_b(z)} = 1/\,\mathrm{e}_b(\bar{z}); \tag{7}$$

pentagon equation:

$$\mathrm{e}_b(\mathsf{p})\,\mathrm{e}_b(\mathsf{q}) = \mathrm{e}_b(\mathsf{q})\,\mathrm{e}_b(\mathsf{p}+\mathsf{q})\,\mathrm{e}_b(\mathsf{p}), \tag{8}$$

if self-adjoint operators p and q in $L^2(\mathbb{R})$ satisfy the Heisenberg commutation relation $[\mathsf{p},\mathsf{q}] = (2\pi\mathsf{i})^{-1}$;

integral analogue of Ramanujan's summation formula:

$$\begin{aligned}\int_{\mathbb{R}} \frac{\mathrm{e}_b(x+u)}{\mathrm{e}_b(x+v)} e^{2\pi\mathrm{i}wx}\,dx \\ = \frac{\mathrm{e}_b(u-v-c_b)\,\mathrm{e}_b(w+c_b)}{\mathrm{e}_b(u-v+w-c_b)} e^{-2\pi\mathrm{i}w(v+c_b)+\mathrm{i}\pi(1-4c_b^2)/12} \\ = \frac{\mathrm{e}_b(v-u-w+c_b)}{\mathrm{e}_b(v-u+c_b)\,\mathrm{e}_b(-w-c_b)} e^{-2\pi\mathrm{i}w(u-c_b)-\mathrm{i}\pi(1-4c_b^2)/12},\end{aligned} \tag{9}$$

where

$$\Im(v+c_b) > 0, \quad \Im(-u+c_b) > 0, \quad \Im(v-u) < \Im w < 0. \tag{10}$$

Restrictions (10) can actually be relaxed by deforming the integration path in the complex x plane, keeping the asymptotic directions of the two ends within the sectors $\pm(|\arg x| - \pi/2) > \arg b$. The enlarged in this way domain for the variables u, v, w in eqn (9) has the form:

$$|\arg(\mathrm{i}z)| < \pi - \arg b, \quad z \in \{w, v-u-w, u-v-2c_b\}. \tag{11}$$

As the matter of fact the pentagon equation (8) is equivalent to the integral Ramanujan formula (9), see [7] for the proof.

3. The basic algebraic system

Introduce an important notation. Let V be a vector space. For any natural $1 \le i \le m$ we define embeddings

$$\iota_i\colon \operatorname{End} V \ni \mathsf{a} \mapsto \mathsf{a}_i = \underbrace{id \otimes \cdots \otimes id}_{i-1 \text{ times}} \otimes \mathsf{a} \otimes id \otimes \cdots \otimes id \in \operatorname{End} V^{\otimes m},$$

ie a stands in the i-th position. If $\mathsf{b} \in \operatorname{End} V^{\otimes k}$ for some $1 \le k \le m$ and $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, m\}$, we write

$$\mathsf{b}_{i_1 i_2 \ldots i_2} \equiv \iota_{i_1} \otimes \iota_{i_2} \otimes \cdots \otimes \iota_{i_k}(\mathsf{b}).$$

Note also that the permutation group $\mathbb{S}_m$ is naturally represented in $V^{\otimes m}$:

$$\mathsf{P}_\sigma(x_1 \otimes \cdots \otimes x_i \otimes \cdots) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i)} \otimes \ldots, \quad \sigma \in \mathbb{S}_m. \tag{12}$$

The projective representation of the mapping class groups of punctured surfaces, associated with the quantum Teichmüller theory, is based on the concrete realization of the following algebraic system of equations on two invertible elements $\mathsf{A} \in \operatorname{End} V$ and $\mathsf{T} \in \operatorname{End} V^{\otimes 2}$,

$$\mathsf{A}^3 = id, \tag{13}$$

$$\mathsf{T}_{12}\mathsf{T}_{13}\mathsf{T}_{23} = \mathsf{T}_{23}\mathsf{T}_{12}, \tag{14}$$

$$\mathsf{A}_1\mathsf{T}_{12}\mathsf{A}_2 = \mathsf{A}_2\mathsf{T}_{21}\mathsf{A}_1, \tag{15}$$

$$\mathsf{T}_{12}\mathsf{A}_1\mathsf{T}_{21} = \zeta\mathsf{A}_1\mathsf{A}_2\mathsf{P}_{(12)}, \tag{16}$$

where ζ is a (nonzero) complex number, $\mathsf{P}_{(12)}$ being defined in eqn (12) with

$$\mathbb{S}_2 \ni \sigma = (12)\colon 1 \mapsto 2 \mapsto 1.$$

3.1. **Useful notation.** In practical calculations it is very convenient to use the following notation which permits to avoid writing explicitly operator A. For any $\mathsf{a} \in \operatorname{End} V$ we denote

$$\mathsf{a}_{\hat{k}} \equiv \mathsf{A}_k \mathsf{a}_k \mathsf{A}_k^{-1}, \quad \mathsf{a}_{\check{k}} \equiv \mathsf{A}_k^{-1} \mathsf{a}_k \mathsf{A}_k. \tag{17}$$

Obviously

$$\mathsf{a}_{\check{\hat{k}}} = \mathsf{a}_{\hat{\check{k}}} = \mathsf{a}_k, \quad \mathsf{a}_{\hat{\hat{k}}} = \mathsf{a}_{\check{k}}, \quad \mathsf{a}_{\check{\check{k}}} = \mathsf{a}_{\hat{k}},$$

where the last two equations follow from eqn (13). Besides, it is useful to denote

$$\mathsf{P}_{(kl\ldots m\hat{k})} \equiv \mathsf{A}_k \mathsf{P}_{(kl\ldots m)}, \quad \mathsf{P}_{(kl\ldots m\check{k})} \equiv \mathsf{A}_k^{-1} \mathsf{P}_{(kl\ldots m)}, \tag{18}$$

where $(kl\ldots m)$ is cyclic permutation,

$$(kl\ldots m)\colon k \mapsto l \mapsto \ldots \mapsto m \mapsto k.$$

Eqns (15), (16) in this notation take very compact form

$$\mathsf{T}_{12} = \mathsf{T}_{\hat{2}\check{1}}, \tag{19}$$

$$\mathsf{T}_{12}\mathsf{T}_{2\hat{1}} = \zeta \mathsf{P}_{(12\hat{1})}. \tag{20}$$

3.2. **Realization through the quantum dilogarithm.** Take

$$V = L^2(\mathbb{R}).$$

Let self-adjoint operators p, q satisfy the Heisenberg commutation relation

$$[\mathsf{p}, \mathsf{q}] = (2\pi \mathsf{i})^{-1}.$$

Then operators

$$\mathsf{A} \equiv e^{-\mathsf{i}\pi/3} e^{\mathsf{i}3\pi\mathsf{q}^2} e^{\mathsf{i}\pi(\mathsf{p}+\mathsf{q})^2} \in L^2(\mathbb{R}), \tag{21}$$

$$\mathsf{T} \equiv e^{\mathsf{i}2\pi\mathsf{p}_1\mathsf{q}_2} (\mathrm{e}_b(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2))^{-1} \in L^2(\mathbb{R}^2), \tag{22}$$

do satisfy equations (13) – (16) with $\zeta = e^{\mathsf{i}\pi c_b^2/3}$. Note that operator A here is unitary and is characterized by the equations (up to a normalization factor)

$$\mathsf{A}\mathsf{q}\mathsf{A}^{-1} = \mathsf{p} - \mathsf{q}, \quad \mathsf{A}\mathsf{p}\mathsf{A}^{-1} = -\mathsf{q},$$

while operator T is unitary if $(1 - |b|)\Im b = 0$.

4. The mapping class group representation

Here we recall how the mapping class groups of punctured surfaces are represented in terms of system (13) – (16).

4.1. **Decorated ideal triangulations.** We call a two-cell in a cell complex (or CW-complex, see [4]) *triangle* if exactly three boundary points of the corresponding two-disk are mapped to the zero-skeleton. We shall also call zero-cells and one-cells *vertices* and *edges*, respectively.

Let Σ be a compact oriented (possibly with boundary) surface with finite non-empty set V_Σ of marked points called also *punctures*.

Definition 1. *A cell complex decomposition of Σ is called* ideal triangulation *if*

- *the zero-skeleton coincides with V_Σ;*
- *all two-cells are triangles.*

Two ideal triangulations are considered equivalent *if there exists an isotopy of Σ fixed on $\partial\Sigma \cup V_\Sigma$ deforming one into another.*

We suppose that Σ admits ideal triangulations (for example, each boundary component must contain at least one marked point). Any ideal triangulation of Σ has one and the same number of triangles n_Σ. Denote $T(\tau)$ the set of triangles in ideal triangulation τ.

Definition 2. *Ideal triangulation τ is called* decorated *if*

- *each triangle is provided by a marked corner;*
- *all triangles are numbered, ie a bijective* numbering *mapping*

$$\bar{\tau}\colon \{1,\dots,n_\Sigma\} \to T(\tau)$$

is fixed.

Graphically (see below) in a triangle we put the corresponding integer inside of it, and asterisk at the marked corner. The set of all decorated ideal triangulations of Σ will be denoted Δ_Σ. For terminological convenience and if no confusion is possible, in what follows we sometimes will use the term triangulation as a substitute for decorated ideal triangulation.

4.1.1. *Action of the permutation group.* The permutation group $\mathbb{S}_{n_\Sigma}$ naturally acts in Δ_Σ from the right,

$$\Delta_\Sigma \times \mathbb{S}_{n_\Sigma} \ni (\tau,\sigma) \mapsto \tau\sigma \in \Delta_\Sigma,$$

by changing the numbering mapping:

$$\overline{\tau\sigma} = \bar{\tau} \circ \sigma.$$

4.1.2. *Changing of a marked corner.* If $\tau \in \Delta_\Sigma$ and $1 \le i \le n_\Sigma$, then triangulation $\rho_i(\tau)$ is obtained from τ by changing the marked corner of triangle $\bar{\tau}(i)$ as is shown in figure 1.

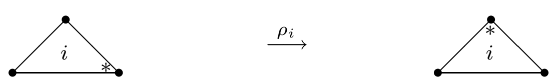

FIGURE 1. Transformation ρ_i changes the marked corner of triangle $\bar{\tau}(i)$.

4.1.3. *The flip transformation.* Let two distinct triangles $\bar{\tau}(i)$, $\bar{\tau}(j)$ have a common edge and their marked corners be as in the lhs of figure 2, so the common edge is a diagonal of a quadrilateral combined of the two triangles. Then triangulation

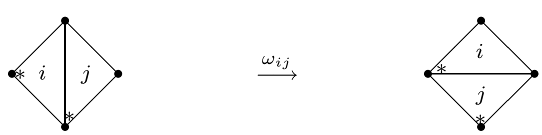

FIGURE 2. The flip transformation ω_{ij} clockwise "rotates" one diagonal of the quadrilateral until it matches another diagonal.

$\omega_{ij}(\tau)$ is obtained from τ by replacing the common edge by the opposite diagonal of the quadrilateral, assigning the numbers and marked corners to new triangles as is shown in the rhs of figure 2. Note that this *flip* transformation ω_{ij} implicitly depends on the triangulation it transforms as at fixed i and j it is not defined on all triangulations.

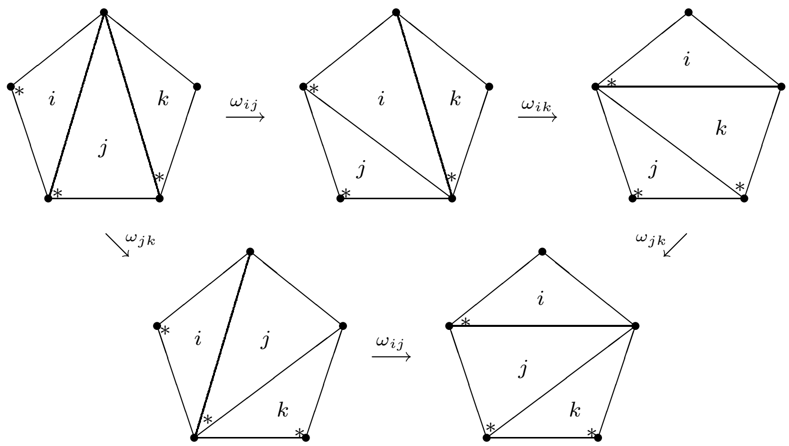

FIGURE 3. Proof of the pentagon equation (24).

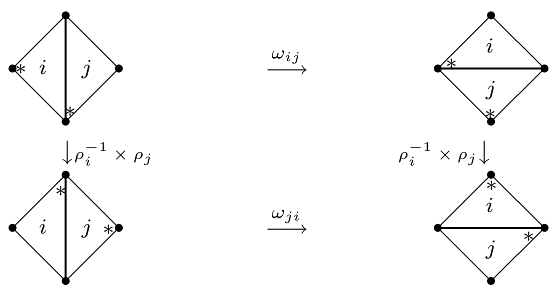

FIGURE 4. Proof of the symmetry relation (25).

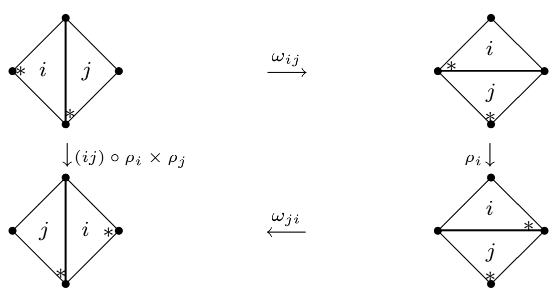

FIGURE 5. Proof of the inversion relation (26).

4.1.4. *The properties.* We have the following properties:

$$\rho_i \circ \rho_i \circ \rho_i = id, \tag{23}$$

$$\omega_{jk} \circ \omega_{ik} \circ \omega_{ij} = \omega_{ij} \circ \omega_{jk}, \tag{24}$$

$$(\rho_i^{-1} \times \rho_j) \circ \omega_{ij} = \omega_{ji} \circ (\rho_i^{-1} \times \rho_j), \tag{25}$$

$$\omega_{ji} \circ \rho_i \circ \omega_{ij} = (ij) \circ (\rho_i \times \rho_j). \tag{26}$$

The first equation is evident since there are only three possibilities to mark a corner in a triangle. The other three equations are proved pictorially in figures 3 – 5.

Any two (decorated ideal) triangulations can be transformed to each other by a (finite) composition of elementary transformations, ie ρ_i, ω_{ij} and permutations. This follows from the known fact that any two ideal triangulations (without decoration) can be transformed one into another by a composition of flips [15], and that

all possible decorations of a fixed ideal triangulation are transitively acted upon by compositions of permutations and ρ_i transformations.

4.2. **Quantum theory.** Suppose we are given a solution to eqns (13) – (16). For each $\tau \in \Delta_\Sigma$ and $1 \le i \le n_\Sigma$ assign

$$\mathsf{F}(\tau, \rho_i(\tau)) \equiv \mathsf{A}_i \in \operatorname{End} V^{n_\Sigma}. \tag{27}$$

Let $i \neq j$ be such that triangles $\bar\tau(i)$ and $\bar\tau(j)$ are as in the lhs of figure 2. Then we put

$$\mathsf{F}(\tau, \omega_{ij}(\tau)) \equiv \mathsf{T}_{ij} \in \operatorname{End} V^{n_\Sigma}. \tag{28}$$

Finally, for any permutation $\sigma \in \mathbb{S}_{n_\Sigma}$ set

$$\mathsf{F}(\tau, \tau\sigma) \equiv \mathsf{P}_\sigma \in \operatorname{End} V^{n_\Sigma},$$

where operator P_σ is defined by eqn (12). Ensured by the consistency of eqns (13) – (16) with eqns (23) – (26), mapping F can be extended to an operator valued function $\mathsf{F}(\tau, \tau')$ on $\Delta_\Sigma \times \Delta_\Sigma$ such that for any $\tau, \tau', \tau'' \in \Delta_\Sigma$

$$\mathsf{F}(\tau, \tau) = 1, \quad \mathsf{F}(\tau, \tau')\mathsf{F}(\tau', \tau'')\mathsf{F}(\tau'', \tau) \in \mathbb{C} - \{0\}. \tag{29}$$

In particular (when $\tau'' = \tau$),

$$\mathsf{F}(\tau, \tau')\mathsf{F}(\tau', \tau) \in \mathbb{C} - \{0\}. \tag{30}$$

As an example, deduce operator $\mathsf{F}(\tau, \omega_{ij}^{-1}(\tau))$. Denoting $\tau' \equiv \omega_{ij}^{-1}(\tau)$ and employing eqn (30) as well as definition (28), we have

$$\mathsf{F}(\tau, \omega_{ij}^{-1}(\tau)) = \mathsf{F}(\omega_{ij}(\tau'), \tau') \simeq (\mathsf{F}(\tau', \omega_{ij}(\tau')))^{-1} = \mathsf{T}_{ij}^{-1}, \tag{31}$$

where we denote by $\simeq$ an equality up to a numerical factor.

The mapping class or modular group[1] $\mathcal{M}_\Sigma$ of Σ naturally acts in Δ_Σ. By construction we have the following invariance property of function F:

$$\mathsf{F}(f(\tau), f(\tau')) = \mathsf{F}(\tau, \tau'), \qquad \forall f \in \mathcal{M}_\Sigma.$$

This enables us to construct a projective representation of $\mathcal{M}_\Sigma$:

$$\mathcal{M}_\Sigma \ni f \mapsto \mathsf{F}(\tau, f(\tau)) \in \operatorname{End} V^{n_\Sigma}.$$

Indeed,

$$\mathsf{F}(\tau, f(\tau))\mathsf{F}(\tau, h(\tau)) = \mathsf{F}(\tau, f(\tau))\mathsf{F}(f(\tau), f(h(\tau))) \simeq \mathsf{F}(\tau, fh(\tau)).$$

Any Dehn twist, at least if it is along a non-separating contour, is equivalent to the Dehn twist of an annulus (with two marked points on the two boundary components) along its only non-contractible loop denoted α in figure 6. As an operator it is given in terms of operator T. Indeed, from figure 6 it follows that $\omega_{12} \circ D_\alpha(\tau) = \tau$. So, using eqn (31), we obtain

$$\mathsf{F}(\tau, D_\alpha(\tau)) = \mathsf{F}(\tau, \omega_{12}^{-1}(\tau)) \simeq \mathsf{T}_{12}^{-1}, \tag{32}$$

where the normalization is to be fixed.

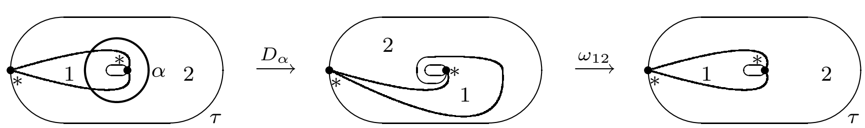

FIGURE 6. The Dehn twist along contour α followed by the flip transformation ω_{12} does not change the initial triangulation.

[1]The group of homeomorphisms identical on the boundary and permuting the interior marked points, factored wrt the connected component of the identical homeomorphism.

5. Diagonalizing the Dehn twist

In this section we work within the quantum Teichmüller theory, ie we consider solution (21), (22) of system (13) – (16). We assume that $(1-|b|)\Im b=0$, so the range of mapping F in this case is given by unitary operators.

Properly normalized quantum Dehn twist of the annulus has the form:

$$\mathsf{D}_\alpha=\zeta^{-6}\mathsf{T}_{12}^{-1}e^{\mathsf{i}2\pi\mathsf{z}_\alpha^2},\quad \mathsf{z}_\alpha=(\mathsf{p}_1+\mathsf{q}_2)/2,\quad \zeta=e^{\mathsf{i}\pi c_b^2/3},$$

where the exponential factor removes the spurious degree of freedom, see [13, 14]. Using formula (22) we rewrite it equivalently

$$\mathsf{D}_\alpha=e^{\mathsf{i}2\pi(\mathsf{q}_\alpha^2-c_b^2)}\,\mathrm{e}_b(\mathsf{p}_\alpha+\mathsf{q}_\alpha),$$

where the self-adjoint operators

$$\mathsf{p}_\alpha\equiv\mathsf{q}_1+\mathsf{p}_2-(\mathsf{q}_2+\mathsf{p}_1)/2,\quad \mathsf{q}_\alpha\equiv(\mathsf{q}_2-\mathsf{p}_1)/2$$

satisfy the Heisenberg commutation relation

$$[\mathsf{p}_\alpha,\mathsf{q}_\alpha]=(2\pi\mathsf{i})^{-1}.$$

To diagonalize D_α, we use the fact that mutually commuting (and conjugate to each other in the case when $|b|=1$) unbounded operators

$$\mathsf{L}_\alpha^\pm\equiv 2\cosh(2\pi b^{\pm1}\mathsf{q}_\alpha)+e^{2\pi b^{\pm1}\mathsf{p}_\alpha}$$

also commute with D_α:

$$[\mathsf{L}_\alpha^-,\mathsf{L}_\alpha^+]=[\mathsf{D}_\alpha,\mathsf{L}_\alpha^\pm]=0.$$

Geometrically operator L_α^+ is nothing else but the quantized geodesic length (hyperbolic cosine thereof) of contour α, see [3, 8].

Let us consider the "coordinate" basis $\langle x|$, $x\in\mathbb{R}$, where q_α is diagonal and p_α is a differentiation operator,

$$\langle x|\mathsf{q}_\alpha=x\langle x|,\quad \langle x|\mathsf{p}_\alpha=\frac{1}{2\pi\mathsf{i}}\frac{\partial}{\partial x}\langle x|,$$

and define one parameter family of vectors:

$$\langle x|\alpha_s\rangle=\frac{\mathrm{e}_b(s+x+c_b-\mathsf{i}0)}{\mathrm{e}_b(s-x-c_b+\mathsf{i}0)}e^{-\mathsf{i}2\pi(x+c_b)s}=\langle x|\alpha_{-s}\rangle,\quad s\in\mathbb{R}. \tag{33}$$

Theorem 1. *Vectors $|\alpha_s\rangle$ are eigenvectors of operators $\mathsf{L}_\alpha^\pm,\mathsf{D}_\alpha$:*

$$\mathsf{L}_\alpha^\pm|\alpha_s\rangle=|\alpha_s\rangle 2\cosh(2\pi b^{\pm1}s),\quad \mathsf{D}_\alpha|\alpha_s\rangle=|\alpha_s\rangle e^{\mathsf{i}2\pi(s^2-c_b^2)}. \tag{34}$$

They are orthogonal to each other:

$$\langle\alpha_r|\alpha_s\rangle=\nu(s)^{-1}\delta(r-s),\quad \nu(s)\equiv 4\sinh(2\pi bs)\sinh(2\pi b^{-1}s), \tag{35}$$

and complete in $L^2(\mathbb{R})$:

$$\int_0^\infty|\alpha_s\rangle\nu(s)\,ds\langle\alpha_s|=1. \tag{36}$$

The continuous spectrum of Dehn twists in quantum Teichmüller theory has also been observed by V. Fock[2]. It is worth noting that measure $\nu(s)ds$ in eqn (36) also appears in the representation theory of the non-compact quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ [18, 19]. In the next section we reveal that this is not accidental.

[2] Private communication.

6. Braiding and R-matrix

Braiding of a disk with two vertices is naturally associated with the contour (isotopy class thereof) surrounding the vertices. Moreover, square of the braiding is Dehn twist along the associated contour.

Figure 7 shows how the braiding of triangulation τ of a disk with two interior and two boundary marked points can be removed by a sequence of elementary transformations:

$$\tau = (13)(24) \circ (\rho_3 \times \rho_1^{-1}) \circ \omega_{14} \circ (\omega_{13} \times \omega_{24}) \circ (\rho_1 \times \omega_{23}) \circ \rho_3^{-1} \circ B_\alpha^{-1}(\tau).$$

Using the construction of subsection 4.2, the corresponding quantum braiding

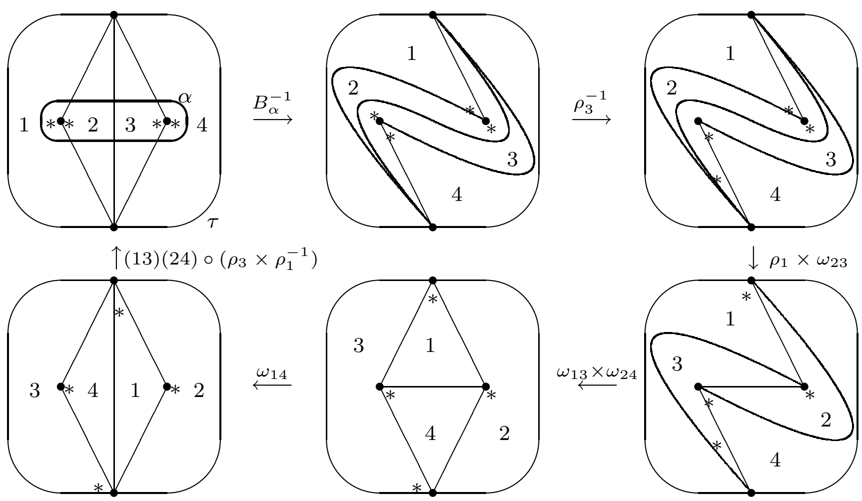

Figure 7. Braiding along contour α followed by the indicated sequence of transformations brings back to the initial triangulation τ.

operator has the form (up to a normalization factor)

$$\mathsf{B}_\alpha \equiv \mathsf{F}(\tau, B_\alpha(\tau)) \simeq \mathsf{P}_{(13)(24)}\mathsf{R}_{1234},$$

with

$$\mathsf{R} = \mathsf{R}_{1234} \equiv \mathsf{A}_1^{-1}\mathsf{A}_3\mathsf{T}_{41}\mathsf{T}_{31}\mathsf{T}_{42}\mathsf{T}_{32}\mathsf{A}_1\mathsf{A}_3^{-1} = \mathsf{T}_{1\check{4}}\mathsf{T}_{13}\mathsf{T}_{42}\mathsf{T}_{\hat{3}2}, \tag{37}$$

where in the second equation we have used notation (17) and eqn (19). As a consequence of relations (13) – (16) operator $\mathsf{R} \in \operatorname{End} V^4$ solves the following Yang–Baxter equation

$$\mathsf{R}_{1234}\mathsf{R}_{1256}\mathsf{R}_{3456} = \mathsf{R}_{3456}\mathsf{R}_{1256}\mathsf{R}_{1234}, \tag{38}$$

and in the case corresponding to realization through the quantum dilogarithm it is in fact (as will be shown below) $\mathcal{U}_q(\mathfrak{sl}(2))$ (more precisely, the modular double thereof [6]) universal R-matrix evaluated at a reducible infinite dimensional representation acting in $L^2(\mathbb{R}^2)$. Note that eqn (37) is a particular case of the realization of universal R-matrices in the Drinfeld doubles of Hopf algebras in terms of solutions of the pentagon equation associated with the Heisenberg doubles [12].

6.1. **Relation to Hopf algebras.** To identify the generators of the quantum group, we first rewrite the R-matrix in the form

$$\mathsf{R} = \operatorname{Ad}(\mathsf{T}_{1\hat{2}}^{-1}\mathsf{T}_{\check{4}3})\mathsf{T}_{\hat{2}\check{4}} = \operatorname{Ad}(\mathsf{A}_2\mathsf{T}_{12}^{-1}\mathsf{A}_4^{-1}\mathsf{T}_{43})\mathsf{T}_{24}, \tag{39}$$

where

$$\mathrm{Ad}(\mathsf{a})\mathsf{x} \equiv \mathsf{a}\mathsf{x}\mathsf{a}^{-1},$$

and we have used the pentagon equation (14). Let now operator T be written as a sum

$$\mathsf{T} = \sum_a \mathsf{e}_a \otimes \mathsf{e}^a,$$

where two operator sets $\{\mathsf{e}_a\}$ and $\{\mathsf{e}^a\}$ are realizations of mutually dual linear bases of two (mutually dual) Hopf sub-algebras A_{T} and A_{T}^* in the associated to T Heisenberg double $H(A_{\mathsf{T}})$. Their co-products are given by the formulae

$$\Delta(\mathsf{e}_a) = \mathrm{Ad}(\mathsf{T}^{-1})(1 \otimes \mathsf{e}_a), \quad \Delta(\mathsf{e}^a) = \mathrm{Ad}(\mathsf{T})(\mathsf{e}^a \otimes 1).$$

From eqn (39) we conclude that the R-matrix is also decomposed into a sum

$$\mathsf{R} = \sum_a \mathsf{E}_a \otimes \mathsf{E}^a$$

with

$$\mathsf{E}_a = \mathrm{Ad}(\mathsf{A}_2\mathsf{T}^{-1})(1 \otimes \mathsf{e}_a) = \mathrm{Ad}(\mathsf{A}_2)\Delta(\mathsf{e}_a), \tag{40}$$

$$\mathsf{E}^a = \mathrm{Ad}(\mathsf{A}_2^{-1}\mathsf{T}_{21})(1 \otimes \mathsf{e}^a) = \mathrm{Ad}(\mathsf{A}_2^{-1})\Delta'(\mathsf{e}^a), \tag{41}$$

where Δ' is the opposite co-product. These are also realizations of the Hopf algebras A_{T} and A_{T}^*, but this time within the Drinfeld double $D(A_{\mathsf{T}})$.

6.2. **Quantum Teichmüller theory and $\mathcal{U}_q(\mathfrak{sl}(2))$.** Here we will see what quantum group and in which representation does correspond to our solution (21), (22).

From eqns (22), (2) we see that all elements $\{\mathsf{e}_a\}$ can be thought to be generated by operators p and $e^{2\pi b^{\pm1}\mathsf{q}}$ with the co-product

$$\Delta(\mathsf{p}) = \mathsf{p}_1 + \mathsf{p}_2, \quad \Delta(e^{2\pi b^{\pm1}\mathsf{q}}) = e^{2\pi b^{\pm1}(\mathsf{q}_1+\mathsf{p}_2)} + e^{2\pi b^{\pm1}\mathsf{q}_2}.$$

Similarly dual elements $\{\mathsf{e}^a\}$ are generated by q and $e^{2\pi b^{\pm1}(\mathsf{p}-\mathsf{q})}$ with the co-product

$$\Delta(\mathsf{q}) = \mathsf{q}_1 + \mathsf{q}_2, \quad \Delta(e^{2\pi b^{\pm1}(\mathsf{p}-\mathsf{q})}) = e^{2\pi b^{\pm1}(\mathsf{p}_1-\mathsf{q}_1-\mathsf{q}_2)} + e^{2\pi b^{\pm1}(\mathsf{p}_2-\mathsf{q}_2)}.$$

From eqn (40) we deduce that operators

$$\mathsf{g}_{12} \equiv \mathsf{p}_1 - \mathsf{q}_2, \quad \mathsf{f}_{12}^{\pm} \equiv e^{2\pi b^{\pm1}(\mathsf{q}_1-\mathsf{q}_2)} + e^{2\pi b^{\pm1}(\mathsf{p}_2-\mathsf{q}_2)} \tag{42}$$

generate the set $\{\mathsf{E}_a\}$, while according to eqn (41) the generators of the dual set $\{\mathsf{E}^a\}$ are the operators

$$\mathsf{q}_1 - \mathsf{p}_2 = -\mathsf{g}_{21}, \quad e^{2\pi b^{\pm1}(\mathsf{q}_2-\mathsf{q}_1)} + e^{2\pi b^{\pm1}(\mathsf{p}_1-\mathsf{q}_1)} = \mathsf{f}_{21}^{\pm}. \tag{43}$$

In what follows we restrict our attention to the sub-algebra corresponding to the positive exponent of parameter b.

Proposition 1. *Operators $\mathsf{f}_{mn} \equiv \mathsf{f}_{mn}^+$, g_{mn} ($mn = 12$ or 21) with the relations*

$$[\mathsf{g}_{12}, \mathsf{g}_{21}] = 0, \quad [\mathsf{g}_{mn}, \mathsf{f}_{mn}] = -\mathrm{i}b\mathsf{f}_{mn}, \quad [\mathsf{g}_{nm}, \mathsf{f}_{mn}] = \mathrm{i}b\mathsf{f}_{mn},$$

$$[\mathsf{f}_{12}, \mathsf{f}_{21}] = (q - q^{-1})(e^{2\pi b\mathsf{g}_{12}} - e^{2\pi b\mathsf{g}_{21}}), \quad q \equiv e^{\mathrm{i}\pi b^2};$$

and the twisted co-product

$$\Delta_\varphi = \mathrm{Ad}(e^{\mathrm{i}\varphi(\mathsf{g}_{21}\otimes\mathsf{g}_{12}-\mathsf{g}_{12}\otimes\mathsf{g}_{21})}) \circ \Delta, \quad \varphi \in \mathbb{R},$$

$$\Delta(\mathsf{g}_{mn}) = \mathsf{g}_{mn} \otimes 1 + 1 \otimes \mathsf{g}_{mn},$$

$$\Delta(\mathsf{f}_{12}) = \mathsf{f}_{12} \otimes e^{2\pi b\mathsf{g}_{12}} + 1 \otimes \mathsf{f}_{12}, \quad \Delta(\mathsf{f}_{21}) = e^{2\pi b\mathsf{g}_{21}} \otimes \mathsf{f}_{21} + \mathsf{f}_{21} \otimes 1.$$

generate a Hopf algebra $\mathcal{G}_\varphi$.

Algebra $\mathcal{G}_{\pi/2}$ is closely related with $\mathcal{U}_q(\mathfrak{sl}(2))$.

Definition 3. [3] *Quantum group $\mathcal{U}_q(\mathfrak{sl}(2))$ is a Hopf algebra with*

- *generators E, F, K, K^{-1};*
- *relations*

$$KE = qEK, \quad KF = q^{-1}FK, \quad [E, F] = -(K^2 - K^{-2})/(q - q^{-1});$$

- *co-product*

$$\Delta(K) = K \otimes K, \quad \Delta(X) = X \otimes K + K^{-1} \otimes X, \quad X = E \text{ or } F.$$

Proposition 2. *There exists a faithful Hopf algebra homomorphism*

$$\eta \colon \mathcal{U}_q(\mathfrak{sl}(2)) \hookrightarrow \mathcal{G}_{\pi/2}$$

such that

$$K \mapsto e^{\pi b(\mathsf{g}_{12} - \mathsf{g}_{21})/2},$$

$$E \mapsto e^{-\pi b(c_b + \mathsf{g}_{21})} \frac{\mathsf{f}_{21}}{q - q^{-1}}, \quad F \mapsto \frac{\mathsf{f}_{12}}{q - q^{-1}} e^{\pi b(c_b - \mathsf{g}_{12})}.$$

The generators of algebra $\mathcal{G}_\varphi$ depend on only three combinations of the Heisenberg operators $\mathsf{p}_i, \mathsf{q}_i$ $(i = 1, 2)$

$$\mathsf{z}_\beta \equiv (\mathsf{p}_1 - \mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2)/2,$$

$$\mathsf{p}_\beta \equiv (\mathsf{p}_1 + \mathsf{q}_1 - \mathsf{p}_2 - \mathsf{q}_2)/2, \quad \mathsf{q}_\beta \equiv (-\mathsf{p}_1 + \mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2)/2$$

which satisfy the commutation relations

$$[\mathsf{z}_\beta, \mathsf{p}_\beta] = [\mathsf{z}_\beta, \mathsf{q}_\beta] = 0, \quad [\mathsf{p}_\beta, \mathsf{q}_\beta] = (2\pi\mathsf{i})^{-1}.$$

Subscript β here refers to the associated to our operators contour around the interior vertex of a triangulated disk (with one interior vertex) in figure 8. In particular, in quantum Teichmüller theory operator z_β describes the spurious degree of freedom associated with contour β. Substituting these definitions into eqns (42), (43) we

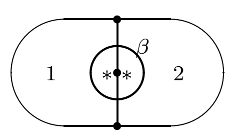

Figure 8. Triangulated disk with one interior vertex.

obtain

$$\mathsf{g}_{12} = \mathsf{z}_\beta + \mathsf{p}_\beta, \quad \mathsf{g}_{21} = \mathsf{z}_\beta - \mathsf{p}_\beta,$$

$$\mathsf{f}_{12} = 2e^{2\pi b \mathsf{q}_\beta} e^{\pi b(\mathsf{z}_\beta + \mathsf{p}_\beta - c_b)} \sinh(\pi b(\mathsf{z}_\beta - \mathsf{p}_\beta + c_b)),$$

$$\mathsf{f}_{21} = -2e^{\pi b(\mathsf{z}_\beta - \mathsf{p}_\beta + c_b)} e^{-2\pi b \mathsf{q}_\beta} \sinh(\pi b(\mathsf{z}_\beta + \mathsf{p}_\beta + c_b)),$$

while R-matrix (37) takes the form

$$\mathsf{R} = e^{\mathsf{i}2\pi(\mathsf{p}_{\beta_1} + \mathsf{z}_{\beta_1})(\mathsf{p}_{\beta_2} - \mathsf{z}_{\beta_2})} \operatorname{Ad}\left(\frac{\mathrm{e}_b(\mathsf{p}_{\beta_1} - \mathsf{z}_{\beta_1})}{\mathrm{e}_b(\mathsf{p}_{\beta_2} + \mathsf{z}_{\beta_2})}\right) (\mathrm{e}_b(\mathsf{q}_{\beta_1} + \mathsf{z}_{\beta_1} - \mathsf{q}_{\beta_2} - \mathsf{p}_{\beta_2}))^{-1}. \tag{44}$$

On the eigenvectors $\langle \xi, x|$ of operators z_β and q_β,

$$\langle \xi, x|\mathsf{z}_\beta = \xi \langle \xi, x|, \quad \langle \xi, x|\mathsf{q}_\beta = x\langle \xi, x|, \quad \langle \xi, x|\mathsf{p}_\beta = \frac{1}{2\pi\mathsf{i}} \frac{\partial}{\partial x} \langle \xi, x|,$$

algebra $\mathcal{G}_\varphi$ is irreducibly represented for each fixed $\xi \in \mathbb{R}$, and these are exactly the representations from Schmüdgen's classification list [20] considered by Ponsot and Teschner in [18, 19].

[3] This definition is that of [19] without concretizing the star-structure, since the latter depends on a solution to the equation $(1 - |b|)\Im b = 0$.

Theorem 2. *If $\Im b = 0$, the following equation holds*

$$\langle \xi, x | \eta(a) = \pi_{\mathsf{i}(\xi - c_b)}(a) \langle \xi, x|, \quad \forall a \in \mathcal{U}_q(\mathfrak{sl}(2)),$$

where representations π_α, $\alpha \in \mathsf{i}(\mathbb{R} - c_b)$, are defined in eqn (12) of [19].

Now it becomes clear the relevance of the result of Ponsot and Teschner on the decomposition of the tensor products of the representations under consideration to the spectral problem of Dehn twists in quantum Teichmüller theory considered in section 5: the decomposition of tensor products of representations into irreducibles is essentially equivalent to the spectral problem of the R-matrix, while square of the latter is nothing else but a Dehn twist. Elaboration of this connection is to be published elsewhere.

7. Conclusion

In this paper we have described the spectrum of Dehn twists (Theorem 1) in the quantum Teichmüller theory and demonstrated appearance of certain infinite dimensional representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2))$ (Theorem 2) studied in [20, 18, 19]. This indicates a relationship between the two approaches to quantum Liouville theory: one through quantization of the Teichmüller spaces and another through representation theory of a non-compact quantum group. The result of [2] on the representation theory of the quantum Lorentz group is also likely to be relevant here. Besides, there should exist direct connections to quantum Liouville theory on a space-time lattice [7] where the non-compact quantum dilogarithm plays the central role too.

We have derived the R-matrix (associated with braidings in the mapping class groups) in terms of the non-compact quantum dilogarithm, formula (44), which first has been suggested by Faddeev in [6] as the universal $\mathcal{U}_q(\mathfrak{sl}(2))$ R-matrix for the corresponding modular double. Note that more general formula (37) directly follows from the canonical embedding of the Drinfeld doubles of Hopf algebras into tensor product of two Heisenberg doubles [12]. Thus the explanation of section 6 can be considered as a geometrical view on the pure algebraic construction of [12]. It is also worth mentioning that this R-matrix is the direct non-compact analogue of the finite dimensional R-matrix introduced in [10] which leads to a specialization of the colored Jones link invariants (polynomials) [16] with the particularly remarkable asymptotic behavior [11]. In this light it would be interesting to study the corresponding non-compact analogue of the Jones invariants.

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